Final Exam — Complex Analysis

Martini Plaza, Monday 26 January 2015, 14:00 - 17:00

Duration: 3 hours

Instructions

1. The test consists of 6 questions; answer all of them.
2. Each question gets 15 points and the number of points for each subquestion is indicated at the beginning of the subquestion. 10 points are "free" and the total number of points is divided by 10. The final grade will be between 1 and 10.
3. The use of books, notes, and calculators is not allowed.

Question 1 (15 points)

Consider the function $f(z) = ze^z$ with z in $\mathbb{C}$.

a. (7 points) Write $f(z)$ in the form $u(x, y) + iv(x, y)$ where $z = x + iy$.

b. (8 points) Use the Cauchy-Riemann equations to show that $f(z)$ is entire.

Question 2 (15 points)

Consider the function $f(z) = ze^{1/z^2}$.

a. (9 points) Find the Laurent series for $f(z)$ in $|z| > 0$.

b. (6 points) What is the type of the singularity of $f(z)$ at 0? Explain your answer.

Question 3 (15 points)

Consider the function

$$f(z) = \frac{e^{-iz}}{z^2 + 9}.$$

a. (6 points) Compute the residue of $f(z)$ at each one of the singularities of the function.

b. (9 points) Evaluate

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 + 9}\, dx.$$

Question 4 (15 points)

a. (7 points) Given the function $f(z) = e^{\sin z}(z - i)^2(z + 2)(z - 3i)$ compute the integral

$$\int_C \frac{f'(z)}{f(z)}\, dz,$$

where C is the positively oriented circular contour with $|z| = 5/2$.

b. (8 points) Use Rouché's theorem to show that the polynomial $P(z) = z^4 - \frac{3}{2}z^3 + 1$ has exactly 4 roots in the disk $|z| < 2$.

Question 5 (15 points)

We denote by $\operatorname{Log} z$ the principal value of the logarithm $\log z$.

a. (6 points) Give without proof the expression for $\operatorname{Log} z$ in terms of the absolute value and the principal argument of z. Where is $\operatorname{Log} z$ analytic?

b. (9 points) Show that the function $g(z) = \operatorname{Log}(-z) + i\pi$ is a branch of $\log z$. Where is $g(z)$ analytic?

Question 6 (15 points)

a. (8 points) Prove that if a function $f(z)$ is analytic inside and on a circle C_R of radius R centered at z_0 and if $|f(z)| \leq M$ for all z on C_R, then

$$|f'(z_0)| \leq \frac{M}{R}.$$

[Hint: recall the generalized Cauchy integral formula]

b. (7 points) Use the estimate of $|f'(z_0)|$ from the previous subquestion to show that a bounded entire function must be constant.

End of the test.